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THE SOLOW-SWAN MODEL WITH A NEGATIVE LABOR GROWTH RATE

HONGJING CHEN and SHAOYONG LAI

School of Economic Mathematics
Southwestern University of Finance and Economics
610074, Chengdu
P. R. China
e-mail: laishaoy@swufe.edu.cn

Abstract

This paper is devoted to the study of the Solow-Swan growth model with a variable population growth rate, which is bounded. The model is classified as a differential equation as the population growth rate is a negative and non-constant number, which represents that the nation has the phenomenon of population aging. Specifically, under certain circumstances, the modified Solow-Swan model could be used to explain the relationship between population aging and wealth gathering.

1. Introduction

In 1956, Solow [5] and Swan [6] developed the classical Solow-Swan model to explain the output of a nation in terms of the level of capital stock, labor force, and technology improvement. This is the origination of neoclassical growth model theory. Usually, the factor of technology is set as an exogenous variable and the labor growth rate is a constant. However, there are some limitations that the model does not explain how or why technological progress occurs. The limitations led to the

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development of endogenous growth theory. There are a lot of research works to investigate the Solow-Swan growth model. Romer [4] investigated that it is the result of intention actions of people, which contribute to technological change. Barro and Martin [1] studied the new classical macroeconomics. Motivated by the desire to further develop the work made by Guerrini [3] in which the population growth rate is positive and depends on time t. In this paper, we investigate a one-sector growth model with a bounded population growth rate, which can be chosen as a negative number. From the study of the model, it will make us comprehensively understand the effects on capital gathering when the population aging happened. The mathematical approaches solving the differential equations are employed to investigate the model, we discussed. Moreover, a specific example is analyzed to support our conclusions, which give helpful explanations in reality. We assume that the labor growth rate is equal to the population growth rate (also, see [1, 3-5]).

2. The Model

Guerrini [3] derived the modified Solow-Swan model

$$\begin{cases} k' = sf(k) - (\delta + n(t))k, \\ k(0) = k_0, \end{cases} \tag{1}$$

where $k' = \frac{dk}{dt}$, $s \in (0, 1)$ is the fraction of output, which is saved, $\delta \in (0, 1)$ is the depreciation rate, and $n(t)$ represents the population growth rate. Moreover, $k = \frac{K}{L}$ is the ratio of capital-labor. Then, $k' = \frac{K'}{L}$ denotes the change of capital-capitol. Guerrini [3] assume that $n(t)$ is positive and bounded, $0 \le n(t) \le n^*$. In this article, provided that $n(t) < 0$ and $0 \le \delta + n(t) \le C$ (C is a constant), we investigate the model (1). If $n(t)$ is a constant, the system (1) is the well-known Solow-Swan model (see [3]) based on the production function $F(K, L)$ with constant return to scale, which can be written as $F(\lambda K, \lambda L) = \lambda F(K, L)$, for all $\lambda > 0$. In fact, Solow set $Q = TF(K, L)$, $q = \frac{Q}{L} = Tf(k)$ (Q represents the total output of a nation, T is technical development, K is total capital, $\dot{L}$ is the gross labor population number, and $q = \frac{Q}{L}$ means the output per person). Solow presumed that the change of T can provide the same output increasing of the margin quantity of K and L. Therefore, we let T be an external variable. We conclude the production function $q = f(k)$, k_0 is an initial value to be selected to study model (1). It explains how the capital increases. The net increase in the stock of capital equals the investment of capital, which comes from the saving of output, less depreciation of capital, $K' = sF(K, L) - \delta K$. Using certain assumptions, Guerrini [3] proved that the solution of system (1) is asymptotically stable. The solution is expressed as hypergeometric functions when the production function is a Cobb-Douglas function and the population growth is similar to the logistic growth law. A conclusion is given in [3] that in the long run, the countries will be stabilized for the same per capita-capital, if their labor growth rate limits are equal. However, in this paper, we will study the model (1) in which the labor growth rate is negative and bounded. Namely, $n^* \leq n(t) \leq -\delta \leq 0$ for all t.

We assume that the Inada conditions presented in [3] are satisfied as follows:

(1) $f(0) = 0$, $f(+\infty) = +\infty$,

(2) $f(k)$ is continuous and differentiable,

(3) $f(k)$ is monotone increasing,

(4) f' is monotone decreasing,

(5) $f'(0) = +\infty$,

(6) $f'(+\infty) = 0$.

These assumptions imply that $k(t) > 0$ and protect the stability of the neoclassical growth model. We continue to consider the modified model (1) under the Inada conditions.

Lemma 1. *If* $n^* \leq n(t) \leq -\delta \leq 0$ *for all* t, *let* $k_i(t)(i = 1, 2)$ *be the solution of the Cauchy problem*

$$\begin{cases} k' = sf(k) - (\delta + n(t))k, \\ k_i(0) = k_{i,0}. \end{cases}$$

If $k_{1,0} < k_{2,0}$, *then* $k_1(t) < k_2(t)$ *for all* t.

Proof. From the Picard's existence theorem ([2]), we know the uniqueness of solutions to the first-order equation with given initial conditions. Using the comparison theorem of differential equations, we get

$$\begin{cases} k_1' = sf(k_1) - (\delta + n(t))k_1, \\ k_1(0) = k_{1,0}, \end{cases} \tag{2}$$

and

$$\begin{cases} k_2' = sf(k_2) - (\delta + n(t))k_2, \\ k_2(0) = k_{2,0}. \end{cases} \tag{3}$$

If $k_{1,0} < k_{2,0}$, then $k_1(t) < k_2(t)$.

Lemma 2. *Let* $k_1(t)$ *be the solution of the system*

$$\begin{cases} k' = sf(k) - (\delta + n_1(t))k, \\ k_0 = k(0), \end{cases}$$

and let $k_2(t)$ *satisfy*

$$\begin{cases} k' = sf(k) - (\delta + n_2(t))k, \\ k_0 = k(0). \end{cases}$$

If $n_1(t) \leq n_2(t)$ *for all t, then* $k_1(t) \geq k_2(t)$ *for all t.*

Proof. This follows from the comparison theorem. If $n_1(t) \le n_2(t)$, then $\delta + n_1(t) \le \delta + n_2(t)$, $-(\delta + n_1(t)) \ge -(\delta + n_2(t))$. We get $k_1(t) \ge k_2(t)$.

Lemma 3. *Let* $k_1(t)$ *be the solution of*

$$\begin{cases} k' = sf(k) - (\delta + n^*)k, \\ k(0) = k_0, \end{cases} \tag{4}$$

$k_2(t)$ *be the solution of*

$$\begin{cases} k' = sf(k) - \delta k, \\ k(0) = k_0, \end{cases} \tag{5}$$

and $k(t)$ *be the solution of problem* (1), *then* $k_1(t) \ge k(t) \ge k_2(t)$.

Proof. It follows from the comparison theorem of differential equations. Using $n^* \le n(t) \le 0$, $\delta + n^* \le \delta + n(t) \le \delta$, we get $k_1(t) \ge k(t) \ge k_2(t)$.

Lemma 4. *Let* $k(t)$ *be the solution of*

$$\begin{cases} k' = sf(k) - (\delta + M)k, \\ k(0) = k_0, \end{cases} \tag{6}$$

where M is a constant.

(i) *If* $M \ge 0$, *Then* $k(t)$ *converges to the unique non-trivial steady state* k^* *of system* (6) *as* $t \to +\infty$.

(ii) *If* $-\delta \le M < 0$, *then* $k(t)$ *converges to the unique non-trivial steady state* k^* *as* $t \to +\infty$.

(iii) *If* $n^* \le M < -\delta$, *then there is a solution for the one order differential equation*

$$k(t) = e^{-\int_0^\tau (\delta+M)d\tau}\left[k_0 + \int_0^\tau f(k)e^{\int_0^\tau (\delta+M)d\tau} d\tau\right]. \tag{7}$$

Proof. The proof of (i) can be found in [5]. In fact, conclusion (i) is used to prove the asymptotic stability of the solution for the Cauchy problem when $M \geq 0$, which is to say that the labor growth rate is positive. The proof of (iii) is a routine argument of differential equations. Here, we only to prove (ii). We create two new functions $z = \frac{sf(k)}{k}$ and $z = \delta + M$. This intersection of the two function k^* characterize the steady state. Considering $-\delta \leq M < 0$ and $\delta + M \geq 0$, we obtain that if $k < k^*$, then $k(t)$ will increase toward k^* and if $k > k^*$, then $k(t)$ will decrease toward k^*.

Remark 1. In fact, Guerrini [3] obtained the following conclusions:

(i) k^* is the unique solution of the equation $sf(k) = (\delta + M)k$.

(ii) The map of $k \to sf(k)/k$ is monotone decreasing.

(iii) The term $sf(k(t))/k(t)$ makes sense if $k(t) > 0$.

Theorem 1. *Let* $k_1(t)$, $k_2(t)$ *be defined as in Lemma* 3. *Let* k_1^*, k_2^* *be the unique non-trivial steady state of the corresponding Cauchy problems. It holds that*

(i) *If* $n^* < -\delta < 0$, *then* $k_1^* > k_2^*$.

(ii) $\lim_{t \to +\infty} k(t) = k_\infty^*$, *the steady state of the Solow-Swan model with* $n = n_\infty$.

Proof. (i) If $k_1^* \leq k_2^*$, we have $\delta = sf(k_2^*)/k_2^* \leq sf(k_1^*)/k_1^* = \delta + n^*$. This is a contradiction. Thus, we know that (i) is valid. The proof of (ii) can be found in [5].

Theorem 2. *All the assumptions in Theorem* 1 *are satisfied. Let* $n(t)$ *be a monotone decreasing function* $(n_\infty = n^*, n(0) = 0)$. *It holds that*

(i) *If* $k_0 \leq k_2^*$ *for all t, then* $k'(t) \geq 0$.

(ii) *If* $k_2^* < k_0 \le k_1^*$, *then there exists* $\tau > 0$ *such that* $k'(t) \le 0$ *for* $t \in (0, \tau]$ *and* $k'(t) \ge 0$ *for* $t \in [\tau, \infty)$.

(iii) *If* $k_1^* < k_0$, *then* $k'(t) \le 0$ *for all t.*

Proof. (i) There exists a right neighborhood $(0, \eta)$ of $t = 0$, where $k'(t) > 0$. In fact, if $k_0 < k_2^*$, from Remark 1, we have $sf(k_0)/k_0 > sf(k_2^*)/k_2^* = \delta + n(0)$, i.e., $k'(0) > 0$. If $k_0 = k_2^*$, then $k'(0) = 0$. From $k'' = sf'(k(t)) - (\delta + n(t))k(t) - n'k(t)$, we have $k(0) = -n'(0)k_0 > 0$. By continuity, we conclude that $k'(t) \ge 0$ for all t. In fact, if there were $t_0 > 0$ such that $k'(t_0) < 0$, then there would be $\tau \in (\eta, t_0)$ such that $k'(\tau) = 0$. Since $k'(\tau) = -n'(\tau)k(\tau) > 0$, this proved (i).

(ii) Let $k_1^* < k_0 \le k_2^*$, Remark 1 implies $k'(0) < 0$. If $k'(t) < 0$ for all t_0, then $sf(k(t))/k(t) < \sigma + n(t)$ for all t. As $t \to +\infty$, we have $\delta + n(t) = \delta = sf(k_2^*)/k_2^* < \delta + n^*$. It is a contradiction. Therefore, there exists $t_1 > 0$ such that $k'(t_1) > 0$. Set $\tau = \inf t > 0 : k'(t) > 0$. Proceeding as in (i), we see that there is no $t > \tau$ such that $k'(t) < 0$. It completes the proof of (ii).

(iii) Let $k_1^* < k_0$, Remark 1 gives

$$\delta + n^* = sf(k_1^*)/k_1^* > sf(k)/k > sf(k_0)/k_0,$$

i.e.,

$$k' = \frac{sf(k)}{k} - (\delta + n(t)).$$

Thus,

$$\delta + n^* > \frac{k'(t)}{k(t)} + \delta + n(t) > \frac{k'(0)}{k(0)} + \delta,$$

$$n^* > \frac{k'(t)}{k(t)} + n(t) > \frac{k'(0)}{k(0)},$$

$$k'(t) < (n^* - n(t))k(t) < 0.$$

Then, we get $k'(t) < 0$ for all t. It finishes the proof of (iii).

It is important to investigate the steady state of the Solow-Swan model. A steady state means that in the long run, if the economy begins to deviate from its steady state, it will gradually moves back to it. So, the long-term outcome will finally achieve the point of k^*. It is used to describe the dynamic process of Solow-Swan model.

3. Examples and Conclusions

With the changes of our society, many countries face the phenomenon of population aging and declining birth rate, which result in that the labor growth rate is negative. This fact would be cause great effects on the output of a nation. A concrete example will be discussed in this section to show the statement.

Let $k(t)$ be the solution of problem (1) in which, we choose $f(k) = k^{\alpha}$, $0 < \alpha < 1$. Let $n(t)$ be a negative constant M_1, and set $k^{\alpha} = u$, we get

$$\frac{du}{dt} = u' = \alpha k^{\alpha-1}k',$$

and

$$\frac{k^{\alpha}}{k} = u^{1-\frac{1}{\alpha}}.$$

Furthermore, we have

$$k' = \frac{u'}{\alpha k^{\alpha-1}} = su - (\delta + M_1)u^{\frac{1}{\alpha}}. \tag{8}$$

From (8), we get

$$\begin{aligned} u' &= [su - (\delta + M_1 u^{\frac{1}{\alpha}})]\alpha u^{1-\frac{1}{\alpha}} \\ &= s\alpha u^{2-\frac{1}{\alpha}} - (\delta + M_1)\alpha u. \end{aligned}$$

Then

$$\frac{u'}{\alpha} = su^{2-\frac{1}{\alpha}} - (\delta + M_1)u, \tag{9}$$

which results in

$$\frac{du}{\alpha[su^{2-\frac{1}{\alpha}} - (\delta + M_1)u]} = dt. \tag{10}$$

Now, we choose $\alpha = \frac{1}{2}$. It follows from (10) that

$$\int \frac{2du}{s-(\delta + M_1)u} = \int dt,$$

$$-\frac{2}{\delta + M_1} \int \frac{d(\delta + M_1)u}{s - (\delta + M_1)u} = \int dt. \tag{11}$$

Using (11) yields

$$-\frac{2}{\delta + M_1} \ln|s - (\delta + M_1)u| = t + c,$$

from which, we have

$$s - (\delta + M_1)u = \pm e^{-\frac{\delta + M_1}{2}(t+c)},$$

$$(\delta + M_1)^2 k = \left(s \mp e^{-\frac{\delta + M_1}{2}(t+c)}\right)^2,$$

$$k = \left[\frac{s \mp e^{-\frac{\delta + M_1}{2}(t+c)}}{\delta + M_1}\right]^2.$$

We assume $\delta + M_1 < 0$ and have

$$\lim_{t \to +\infty} e^{-\frac{\delta + M_1}{2}(t+c)} = +\infty.$$

Therefore,

$$\lim_{t \to +\infty} k = +\infty, \quad \lim_{t \to +\infty} f(k) = \lim_{t \to +\infty} k^{\frac{1}{2}} = +\infty. \tag{12}$$

From (12), we get the conclusion that when the negative population growth rate M_1 satisfies $M_1 + \delta < 0$ in which δ is the depreciation rate, the capital will be concentrated. As a nation has the problem of population aging and declining of birth rate, it would directly cause the labor growth rate to be a negative number. From the viewpoint of mathematics, we have the conclusion that the wealth of a nation would increase rapidly, while the capita-capital is increasing by considering the expression (12). But the reality is not so optimistic as we supposed. In fact, the population aging and declining of birth rate usually cause a series of problems, such as wealth gap, more need for society welfare and so on, all of which will have bad effect on the output of a nation. Therefore, our society should keep the labor growth rate in a safe range so as to create best opportunity for the economy development.

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